Series solutions to the cauchy problem for partial differential equations

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Abstract. The method of separation of variables can be used to solve many separable linear partial differential equations (LPDEs). Moreover, variable separation solutions usually are some trigonometric series. In the paper, base on some ideas of this method, we introduce a new technique to solve the Cauchy problem for some LPDEs with the initial conditions consisting of some trigonometric series, power series and exponential series. Then many LPDEs which are not separable are solved, such as some second order elliptic equations, Stokes equations and so on. In addition, the solutions of them can be expressed by trigonometric series, power series or exponential series. Moreover, by using power amd exponential series and an iterative method, we can solve many LPDEs and nonlinear PDEs for the first time.

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1 Introduction

The method of separation of variables (also known as the Fourier method) is one of the oldest and most widely used techniques for solving linear partial differential equations (LPDEs) [1]. Moreover, variable separation solutions are trigonometric series in general. However, this method can not be used to deal with the LPDEs which are not separable. In addition, variable separation solutions usually could not be expressed by power series and exponential series.

For simplicity, we write

$$T_1=\sum_{j=1}^{n}a_j\frac{\mathrm{d}^{2j}}{\mathrm{d}x^{2j}};$$
$$T_2=\sum_{j=1}^{n}a_j\frac{\mathrm{d}^{j}}{\mathrm{d}x^{j}};$$
$$T_3=\sum_{j=1}^{n}a_jx^j\frac{\mathrm{d}^{j}}{\mathrm{d}x^{j}}.$$

Some keys in solving many LPDEs (such as the wave equation) by using the method of separation of variables are:

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(A) The functions in the boundary conditions of these LPDEs can be expressed by trigonometric series (Fourier series).

(B) Stretching transformation (eigenvalues and eigenvectors):

$$T_1 \sin kx = \left(\sum_{j=1}^{n}(-1)^j a_j k^{2j}\right)\sin kx,\ \ k=1,2,\cdots;$$
$$T_1 \cos kx = \left(\sum_{j=1}^{n}(-1)^j a_j k^{2j}\right)\cos kx,\ \ k=0,1,2,\cdots;$$
$$T_2 e^{ikx} = \left(\sum_{j=1}^{n} a_j (ik)^j\right) e^{ikx},\ \ k=0,\pm 1,\pm 2,\cdots.$$

Our work's motivation partly comes from the following ideas:

(A′) There are some functions which can be expressed by power series or exponential series (Taylor series). For examples:

$$\frac{1}{1-x} = \sum_{k=0}^{+\infty} x^k,\ \ x\in(-1,1);$$
$$\frac{1}{1-e^x} = \sum_{k=0}^{+\infty} e^{kx},\ \ x\in(-\infty,0).$$

(B′) Stretching transformation (eigenvalues and eigenvectors):

$$T_2 e^{kx} = \left(\sum_{j=1}^{n} a_j k^j\right) e^{kx},\ \ k=0,1,2,\cdots;$$
$$T_3 x^k = \left(\sum_{j=1}^{n} a_j \prod_{m=0}^{j-1}(k-m)\right) x^k,\ \ k=0,1,2,\cdots.$$

Then base on (A), (B), (A′), (B′), the undetermined coefficient method and the superposition principle for the solution of LPDEs, we introduce a new technique, by using which we can solve the cauchy problem for many LPDEs even if they are not separable, such as some second order elliptic equation, Stokes equations and so on. Moreover, the solutions of them can be trigonometric series, power series or exponential series.

Let $\Lambda_1 = \{e^{\lambda kx}\}_{k=0}^{+\infty}$, $\Lambda_2 = \{x^{\mu k}\}_{k=0}^{+\infty}$ where $\lambda,\mu\in\mathbb{R}\setminus\{0\}$. Then for any $m_1,m_2=0,1,2,\cdots$, we have

$$\begin{cases} e^{\lambda m_1 x} e^{\lambda m_2 x} = e^{\lambda(m_1+m_2)x}\in\Lambda_1, \\ x^{\mu m_1} x^{\mu m_2} = x^{\mu(m_1+m_2)}\in\Lambda_2, \\ m_1+m_2\geq \max\{m_1,m_2\}. \end{cases} \tag{1.1}$$

Then by using an iterative method with respect to (1.1), we can solve many LPDEs and nonlinear PDEs (NPDEs) for the first time.

2 Series solutions to the cauchy problem for some LPDEs

Notation

$\mathbb{R}$ − the real numbers.
$\mathbb{C}$ − the complex numbers.
$e^f = \exp(f)$.
$\mathbb{R}^n = \{(r_1,\cdots,r_n)\,|\,r_j\in\mathbb{R},\ 1\leq j\leq n\}$.
$\mathbb{Z}^n = \{(k_1,\cdots,k_n)\,|\,k_j=0,\pm1,\pm2,\cdots,\ 1\leq j\leq n\}$.
$\mathbb{N}^n = \{(k_1,\cdots,k_n)\,|\,k_j=0,1,2,\cdots,\ 1\leq j\leq n\}$.
$\mathbb{N}_+^n = \{(k_1,\cdots,k_n)\,|\,k_j=1,2,\cdots,\ 1\leq j\leq n\}$.

$$\sum_{k=(k_1,\cdots,k_n)\in\mathbb{Z}^n} a_k = \sum_{m=0}^{+\infty}\sum_{|k|=m} a_k, \qquad |k| = \sum_{j=1}^{n}|k_j|.$$

Let $\Gamma=(\Gamma_{pq})_{n\times n}$ be an $n\times n$ matrix differential operator, and let

$$\Gamma_{pq}=\sum_{h=0}^{m_{pq}}A_{pqh}(t)\partial_t^h\sum_{j=1}^{w_{pqh}}B_{pqhj}(x)\partial_x^{\alpha_{pqhj}},$$

where $1\le p,q\le n$, $m_{pq}\in\mathbb{N}$, $w_{pqh}\in\mathbb{N}_+$, $\alpha_{pqhj}\in\mathbb{N}^n$, $x\in\Omega\subseteq\mathbb{R}^n$, $A_{pqh}(t)\in C[0,T]$, $B_{pqhj}(x)\in C(\Omega)$. In this section, we consider the Cauchy problem for the following LPDEs:

$$\begin{cases}\Gamma u(x,t)=f(x,t),\ \ x\in\Omega,\ 0\le t\le T, & (2.1)\\ \partial_t^h u_q|_{t=0}=\sum\limits_{k\in\Lambda}r_{qhk}\xi_k\in C(\Omega),\ \ 1\le q\le n,\ 0\le h\le m_q-1, & (2.2)\\ f_j=\sum\limits_{k\in\Lambda}\xi_k Z_{kj}(t)\in C(\Omega\oplus[0,T]),\ \ 1\le j\le n, & (2.3)\end{cases}$$

where $u=(u_1,\cdots,u_n)^T$, $f=(f_1,\cdots,f_n)^T$, $\xi_k\in C(\Omega)$, $k\in\wedge\subseteq\mathbb{N}^n$, and $m_q=\max\limits_{1\le p\le n}m_{pq}$, $1\le q\le n$.

Definition 2.1. We say Eq. (2.1)-(2.3) fulfils the Fourier-Taylor conditions, which we shall denote by $u(x,t)\in FT(\Omega\oplus[0,T])$, $\{\xi_k\}_{k\in\Lambda}$, if for any $1\le p,q\le n$, $0\le h\le m_{pq}$, $1\le j\le w_{pqh}$, there exists a sequence $\{l_{pqhjk}\}_{k\in\Lambda}\subseteq\mathbb{C}$ such that

$$B_{pqhj}(x)D^{\alpha_{pqhj}}\xi_k=l_{pqhjk}\xi_k,\ \ k\in\Lambda.$$

Next we solve Eq. (2.1)-(2.3) when $u(x,t)\in FT(U_{n,t})$, $\{\xi_k\}_{k\in\Lambda}$ holds. We let

$$u(x,t)=\sum_{k\in\Lambda}\xi_k T_k(t), \tag{2.4}$$

where $T_k(t)=(T_{k1}(t),\cdots,T_{kn}(t))^T$. Suppose that the following conditions hold:

$$\begin{cases}u=\sum\limits_{k\in\Lambda}\xi_k T_k(t)\in C(\Omega\oplus[0,T]),\\ \\ \partial_t^h\partial_x^{\alpha_{pqhj}}u=\sum\limits_{k\in\Lambda}T_k^{(h)}(t)\partial_x^{\alpha_{pqhj}}\xi_k\in C(\Omega\oplus[0,T]),\ \ 1\le p,q\le n,\ 0\le h\le m_{pq},\ 1\le j\le w_{pqh}.\end{cases} \tag{2.5}$$

Then by substituting the series (2.5) into Eq. (2.1)-(2.3) we have

$$\begin{cases}\sum\limits_{k\in\Lambda}\xi_k\left(\sum\limits_{1\le q\le n,\ 0\le h\le m_{pq},\ 1\le j\le w_{pqh}}l_{pqhjk}A_{pqh}(t)T_{kq}^{(h)}(t)-Z_{kp}(t)\right)=0,\ \ 1\le p\le n,\\ \partial_t^h u_q|_{t=0}=\sum\limits_{k\in\Lambda}T_{kq}^{(h)}(0)\xi_k=\sum\limits_{k\in\Lambda}r_{qhk}\xi_k,\ \ 1\le q\le n,\ 0\le h\le m_q-1,\end{cases}$$

For every $k\in\Lambda$, let

$$\begin{cases}\sum\limits_{1\le q\le n,\ 0\le h\le m_{pq},\ 1\le j\le w_{pqh}}l_{pqhjk}A_{pqh}(t)T_{kq}^{(h)}(t)-Z_{kp}(t)=0,\ \ 1\le p\le n,\\ T_{kq}^{(h)}(0)=r_{qhk},\qquad 1\le q\le n,\ 0\le h\le m_q-1.\end{cases} \tag{2.6}$$

This is a Cauchy problem for an ODEs, so we may get $T_{kq}(t)$, $1\le q\le n$, $k\in\Lambda$.

Definition 2.2. We call the series (2.4) a formal solution of Eq. (2.1)-(2.3) w.r.t. (with respect to) $\{\xi_k\}_{k\in\Lambda}$.

Theorem 2.3. If $u(x,t) \in FT(\Omega \oplus [0,T])$, $\{\xi_k\}_{k\in\Lambda}$, and if the solution of Eq. (2.6) exists and is unique for every $k \in \Lambda$, then the formal solution of Eq. (2.1)-(2.3) w.r.t. $\{\xi_k\}_{k\in\Lambda}$ exists and is unique.

Theorem 2.4. Suppose that the series (2.4) is a formal solution of Eq. (2.1)-(2.3) w.r.t. $\{\xi_k\}_{k\in\Lambda}$. If it satisfies the conditions (2.5), then it is a solution of Eq. (2.1)-(2.3).

Clearly if Λ is a finite set, then the conditions (2.5) hold. So we have:

Theorem 2.5. If Λ is a finite set, then a formal solution of Eq. (2.1)-(2.3) w.r.t. $\{\xi_k\}_{k\in\Lambda}$ is a solution.

Next we solve a well known partial differential equation by using the above technique. The result we obtain is exactly the same as the variable separation solutions. However, our technique is more simple and intuitive.

Example 2.6. (Wave Equation [2]).

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0, \ \ 0 \le x \le l, \ t \ge 0, \ a \in \mathbb{R} \setminus \{0\}, & (2.7) \\ u(x,0) = \sum\limits_{k\in\mathbb{N}_+} A_k \sin\dfrac{k\pi x}{l}, \ \ u_t(x,0) = \sum\limits_{k\in\mathbb{N}_+} B_k \sin\dfrac{k\pi x}{l}, & (2.8) \\ u(0,t) = u(l,t) = 0. & (2.9) \end{cases}$$

If we delete the condition (2.9), then $u(x,t) \in FT([0,l] \oplus [0,+\infty))$, $\left\{\sin\frac{k\pi x}{l}\right\}_{k\in\mathbb{N}_+}$. Next we set

$$u(x,t) = \sum_{k\in\mathbb{N}_+} T_k(t) \sin\frac{k\pi x}{l}. \tag{2.10}$$

It satisfies (2.9). Suppose that the series (2.10) satisfies the following conditions:

$$\begin{cases} u = \sum\limits_{k\in\mathbb{N}_+} T_k(t)\sin\frac{k\pi x}{l} \in C([0,l]\oplus[0,+\infty)), \\ u_{xx} = \sum\limits_{k\in\mathbb{N}_+} T_k(t)\left(\sin\frac{k\pi x}{l}\right)'' \in C([0,l]\oplus[0,+\infty)), \\ u_{tt} = \sum\limits_{k\in\mathbb{N}_+} T_k''(t)\sin\frac{k\pi x}{l} \in C([0,l]\oplus[0,+\infty)). \end{cases} \tag{2.11}$$

Then by substituting the series (2.11) into Eq. (2.7)-(2.9) we have

$$\begin{cases} \sum\limits_{k\in\mathbb{N}_+} \left(T_k'' + \left(\frac{ak\pi}{l}\right)^2 T_k\right)\sin\frac{k\pi x}{l} = 0, \\ u(x,0) = \sum\limits_{k\in\mathbb{N}_+} T_k(0)\sin\frac{k\pi x}{l} = \sum\limits_{k\in\mathbb{N}_+} A_k\sin\frac{k\pi x}{l}, \\ u_t(x,0) = \sum\limits_{k\in\mathbb{N}_+} T_k'(0)\sin\frac{k\pi x}{l} = \sum\limits_{k\in\mathbb{N}_+} B_k\sin\frac{k\pi x}{l}. \end{cases}$$

Next for any $k \in \mathbb{N}_+$, we let

$$\begin{cases} T_k'' + \left(\frac{ak\pi}{l}\right)^2 T_k = 0, \\ T_k(0) = A_k, \\ T_k'(0) = B_k. \end{cases}$$

Then we have

$$T_k(t)=A_k\cos\tfrac{ak\pi}{l}t+\tfrac{l}{ak\pi}B_k\sin\tfrac{ak\pi}{l}t,\quad k\in\mathbb{N}_+.$$

So the formal solution of Eq. (2.7)-(2.9) w.r.t. $\left\{\sin\frac{k\pi x}{l}\right\}_{k\in\mathbb{N}_+}$ is:

$$u(x,t)=\sum_{k\in\mathbb{N}_+}\left(A_k\cos\tfrac{ak\pi}{l}t+\tfrac{l}{ak\pi}B_k\sin\tfrac{ak\pi}{l}t\right)\sin\tfrac{k\pi x}{l}, \tag{2.12}$$

Theorem 2.7. If

$$\sum_{k\in\mathbb{N}_+}k^2|A_k|+k|B_k|<+\infty, \tag{2.13}$$

then the series (2.12) is a solution of Eq. (2.7)-(2.9).

Proof The inequality (2.13) implies that the series (2.12) satisfies the conditions (2.11), so it is a solution of Eq. (2.7)-(2.9) by Theorem 2.4.

The following PDEs could not be dealt with by using the method of separation of variables, one reason is that they are not separable. However, we can get the exact solutions of them.

Example 2.8. (Second order hyperbolic equation)

$$\begin{cases} u_{tt}+au_{xt}+bu_{xx}=0,\\ a,b\in\mathbb{R},\ \Delta=a^2-4b>0,\ x\in\mathbb{R},\ t\geq 0,\\ u(x,0)=\sum\limits_{k\in\mathbb{N}}A_k\cos\frac{k\pi x}{l},\ u_t(x,0)=\sum\limits_{k\in\mathbb{N}_+}B_k\sin\frac{k\pi x}{l}. \end{cases} \tag{2.14}$$

We let

$$A'_k=\begin{cases}\frac{A_k}{2},\ k\in\mathbb{N}_+;\\ A_0,\ k=0;\\ \frac{A_{-k}}{2},\ -k\in\mathbb{N}_+;\end{cases}\qquad B'_k=\begin{cases}\frac{B_k}{2i},\ k\in\mathbb{N}_+;\\ \frac{-B_{-k}}{2i},\ -k\in\mathbb{N}_+.\end{cases}$$

Then we have

$$\begin{cases} u(x,0)=\sum\limits_{k\in\mathbb{N}}A_k\cos\frac{k\pi x}{l}=\sum\limits_{k\in\mathbb{Z}}A'_ke^{\frac{ik\pi x}{l}};\\ u_t(x,0)=\sum\limits_{k\in\mathbb{N}_+}B_k\sin\frac{k\pi x}{l}=\sum\limits_{k\in\mathbb{Z}\setminus\{0\}}B'_ke^{\frac{ik\pi x}{l}}.\end{cases}$$

Thus $u(x,t)\in FT(\mathbb{R}\oplus[0,\infty)),\ \left\{e^{\frac{ik\pi x}{l}}\right\}_{k\in\mathbb{Z}}$. So we let

$$u(x,t)=\sum_{k\in\mathbb{Z}}T_k(t)e^{\frac{ik\pi x}{l}}. \tag{2.15}$$

Suppose that the series (2.15) satisfies the following conditions:

$$\begin{cases} \sum\limits_{k\in\mathbb{Z}}T_k(t)e^{\frac{ik\pi x}{l}}\in C(\mathbb{R}\oplus[0,\infty)),\\ \frac{\partial^2}{\partial x^2}\sum\limits_{k\in\mathbb{Z}}T_k(t)e^{\frac{ik\pi x}{l}}=\sum\limits_{k\in\mathbb{Z}}T_k(t)\left(e^{\frac{ik\pi x}{l}}\right)''\in C(\mathbb{R}\oplus[0,\infty)),\\ \frac{\partial^2}{\partial x\partial t}\sum\limits_{k\in\mathbb{Z}}T_k(t)e^{\frac{ik\pi x}{l}}=\sum\limits_{k\in\mathbb{Z}}T'_k(t)\left(e^{\frac{ik\pi x}{l}}\right)'\in C(\mathbb{R}\oplus[0,\infty)),\\ \frac{\partial^2}{\partial t^2}\sum\limits_{k\in\mathbb{Z}}T_k(t)e^{\frac{ik\pi x}{l}}=\sum\limits_{k\in\mathbb{Z}}T''_k(t)e^{\frac{ik\pi x}{l}}\in C(\mathbb{R}\oplus[0,\infty)).\end{cases} \tag{2.16}$$

Then by substituting the series (2.16) into Eq. (2.14) we have

$$\begin{cases} \sum\limits_{k\in\mathbb{Z}} \left(T_k''+\frac{ik\pi}{l}aT_k'-(\frac{k\pi}{l})^2bT_k\right)e^{\frac{ik\pi x}{l}}=0, \\ u(x,0)=\sum\limits_{k\in\mathbb{Z}} T_k(0)e^{\frac{ik\pi x}{l}}=\sum\limits_{k\in\mathbb{Z}} A_k'e^{\frac{ik\pi x}{l}}, \\ u_t(x,0)=\sum\limits_{k\in\mathbb{Z}} T_k'(0)e^{\frac{ik\pi x}{l}}=\sum\limits_{k\in\mathbb{Z}\setminus\{0\}} B_k'e^{\frac{ik\pi x}{l}}. \end{cases}$$

For any $k\in\mathbb{Z}$, let

$$\begin{cases} T_k''+\frac{ik\pi}{l}aT_k'-\left(\frac{k\pi}{l}\right)^2bT_k=0, \\ T_k(0)=A_k', \\ T_k'(0)=B_k', \end{cases}$$

where $B_0'=0$. Then we get the formal solution of Eq. (2.14) w.r.t. $\left\{e^{\frac{ik\pi x}{l}}\right\}_{k\in\mathbb{Z}}$:

$$u=A_0+\sum_{k\in\mathbb{N}_+}\left[\frac{\left(\sqrt{\Delta}+a\right)k\pi A_k-2lB_k}{2k\pi\sqrt{\Delta}}\cosh_{1k}+\frac{\left(\sqrt{\Delta}-a\right)k\pi A_k+2lB_k}{2k\pi\sqrt{\Delta}}\cosh_{2k}\right], \tag{2.17}$$

where

$$h_{1k}=\frac{-a+\sqrt{\Delta}}{2}\frac{k\pi}{l}t+\frac{k\pi}{l}x,\ \ h_{2k}=\frac{-a-\sqrt{\Delta}}{2}\frac{k\pi}{l}t+\frac{k\pi}{l}x,\quad k\in\mathbb{N}_+.$$

Similar as Theorem 2.7, we have

Theorem 2.9. If

$$\sum_{k\in\mathbb{N}_+} k^2|A_k|+k|B_k|<+\infty,$$

then the series (2.17) is a solution of Eq. (2.14).

Example 2.10. (Second order elliptic equation)

$$\begin{cases} u_{tt}+au_{xt}+bu_{xx}=0,\ a,b>0,\ \Delta=a^2-4b<0,\ x\in\mathbb{R},\ t\geq 0, \\ u(x,0)=\cos e^{2x},\ u_t(x,0)=\sin e^{2x}. \end{cases} \tag{2.18}$$

Note that

$$u(x,0)=\cos e^{2x}=\sum_{k\in\mathbb{N}}(-1)^k\frac{e^{4kx}}{(2k)!},$$
$$u_t(x,0)=\sin e^{2x}=\sum_{k\in\mathbb{N}_+}(-1)^{k-1}\frac{e^{2(2k-1)x}}{(2k-1)!}.$$

So we have $u(x,t)\in FT(\mathbb{R}\oplus[0,\infty))$, $\{e^{2kx}\}_{k\in\mathbb{N}}$. We can get the formal solution of Eq. (2.18) w.r.t. $\{e^{2kx}\}_{k\in\mathbb{N}}$:

$$\begin{aligned} u(x,t)=1+\sum_{k\in\mathbb{N}_+}&\frac{(-1)^k\exp(2k(-at+2x))}{(2k)!}\left(\cos 2k\sqrt{-\Delta}t+\frac{a}{\sqrt{-\Delta}}\sin 2k\sqrt{-\Delta}t\right) \\ &+\frac{(-1)^{k-1}\exp((2k-1)(-at+2x))}{(2k-1)!(2k-1)\sqrt{-\Delta}}\sin(2k-1)\sqrt{-\Delta}t. \end{aligned} \tag{2.19}$$

It's easy to prove that the series (2.19) is a solution of Eq. (2.18).

Example 2.11.

$$\left\{\begin{array}{l} u_t - t(y-3)u_{xxy} = 0,\ (x,y) \in \Omega \subseteq \mathbb{R}^2,\ t \geq 0, \\ u(x,y,0) = x^3(x-\pi/2)^3 \sin(y-3)^{\frac{3}{5}}. \end{array}\right. \tag{2.20}$$

Note that

$$x^3(x-\pi/2)^3 \sin(y-3)^{\frac{3}{5}} = \sum_{(k,m)\in\mathbb{N}_+^2} A_{km}(y-3)^{\frac{3(2m-1)}{5}} \sin 2kx,$$

where

$$A_{km} = \frac{(-1)^{m-1}}{(2m-1)!}\frac{4}{\pi}\int_0^{\frac{\pi}{2}} x^3(x-\pi/2)^3 \sin 2kx \mathrm{d}x, \quad (k,m) \in \mathbb{N}_+^2.$$

So $u(x,y,t) \in FT(\Omega \oplus [0,+\infty))$, $\left\{(y-3)^{\frac{3(2m-1)}{5}} \sin 2kx\right\}_{(k,m)\in\mathbb{N}_+^2}$. We can get the formal solution of Eq. (2.20) w.r.t. $\left\{(y-3)^{\frac{3(2m-1)}{5}} \sin 2kx\right\}_{(k,m)\in\mathbb{N}_+^2}$:

$$u(x,y,t) = \sum_{(k,m)\in\mathbb{N}_+^2} A_{km} \exp\left(-\frac{6}{5}(2m-1)k^2t^2\right)(y-3)^{\frac{3(2m-1)}{5}} \sin 2kx. \tag{2.21}$$

Moreover, we can prove that

$$\sum_{k\in\mathbb{N}_+} \left|\frac{4k^2}{\pi}\int_0^{\frac{\pi}{2}} x^3(x-\pi/2)^3 \sin 2kx \mathrm{d}x\right| < +\infty, \quad m \in \mathbb{N}_+.$$

So the series (2.21) is a solution of Eq. (2.20).

Example 2.12. (Stokes Equations [3]- [5]).

$$\left\{\begin{array}{l} u_{jt} - \nu \sum\limits_{m=1}^{3} u_{jx_mx_m} + p_{x_j} = f_j(x,t),\ \ j=1,2,3, \\ u_{1x_1} + u_{2x_2} + u_{3x_3} = 0, \quad\ \ t \geq 0,\ x = (x_1,x_2,x_3) \in \mathbb{R}^3, \\ u_j(x,0) = \sum\limits_{k\in\mathbb{Z}^3} A_{jk}\varphi_k,\ \ j=1,2,3, \\ f_j(x,t) = \sum\limits_{k\in\in\mathbb{Z}^3} B_{jk}(t)\varphi_k,\ \ j=1,2,3, \end{array}\right. \tag{2.22}$$

where $\varphi_k = \exp(i\lambda_1k_1x_1 + i\lambda_2k_2x_2 + i\lambda_3k_3x_3)$, $\lambda_j \in \mathbb{R}\setminus\{0\}$, $j=1,2,3$, $\nu \geq 0$.

Obviously $(u_1,u_2,u_3,p)^T \in FT(\mathbb{R}^3 \oplus [0,+\infty))$, $\{\varphi_k\}_{k\in\mathbb{Z}^3}$. So we let

$$\left\{\begin{array}{l} u_j(x,t) = \sum\limits_{k\in\mathbb{Z}^3} T_{jk}(t)\varphi_k,\ \ j=1,2,3; \\ p(x,t) = \sum\limits_{k\in\mathbb{Z}^3} T_{4k}(t)\varphi_k. \end{array}\right. \tag{2.23}$$

Suppose that the series (2.23) satisfy the following conditions:

$$\left\{\begin{array}{ll} u_j = \sum\limits_{k\in\mathbb{Z}^3} T_{jk}(t)\varphi_k \in C(\mathbb{R}^3 \oplus [0,+\infty)), \quad j=1,2,3, & (2.24) \\ p = \sum\limits_{k\in\mathbb{Z}^3} T_{4k}(t)\varphi_k \in C(\mathbb{R}^3 \oplus [0,+\infty)), & (2.25) \\ u_{jt} = \sum\limits_{k\in\mathbb{Z}^3} T'_{jk}(t)\varphi_k \in C(\mathbb{R}^3 \oplus [0,+\infty)), \quad j=1,2,3, & (2.26) \\ u_{jx_mx_m} = \sum\limits_{k\in\mathbb{Z}^3} -(\lambda_mk_m)^2 T_{jk}(t)\varphi_k \in C(\mathbb{R}^3 \oplus [0,+\infty)), \quad m,j=1,2,3, & (2.27) \\ p_{x_j} = \sum\limits_{k\in\mathbb{Z}^3} i\lambda_jk_jT_{4k}(t)\varphi_k \in C(\mathbb{R}^3 \oplus [0,+\infty)), \quad j=1,2,3. & (2.28) \end{array}\right.$$

By substituting the series (2.24)-(2.28) into Eq. (2.22) we get

$$\begin{cases} \sum\limits_{k\in\mathbb{Z}^3} [T'_{jk}+\sum\limits_{m=1}^{3}\nu(\lambda_m k_m)^2 T_{jk}+i\lambda_j k_j T_{4k}-B_{jk}]\varphi_k=0, \ \ j=1,2,3, \\ \sum\limits_{k\in\mathbb{Z}^3}(i\lambda_1 k_1 T_{1k}+i\lambda_2 k_2 T_{2k}+i\lambda_3 k_3 T_{3k})\varphi_k=0, \\ u_j(x,0)=\sum\limits_{k\in\mathbb{Z}^3} A_{jk}\varphi_k=\sum\limits_{k\in\mathbb{Z}^3} T_{jk}(0)\varphi_k, \ \ j=1,2,3. \end{cases}$$

For any $k\in\mathbb{Z}^3$, we let

$$\begin{cases} T'_{jk}+\sum\limits_{m=1}^{3}\nu(\lambda_m k_m)^2 T_{jk}+i\lambda_j k_j T_{4k}-B_{jk}=0, \ \ j=1,2,3, \\ \lambda_1 k_1 T_{1k}+\lambda_2 k_2 T_{2k}+\lambda_3 k_3 T_{3k}=0, \\ T_{jk}(0)=A_{jk}, \qquad j=1,2,3. \end{cases} \tag{2.29}$$

For every $j=1,2,3$, the first equation in Eq. (2.29) is multiplied by $\lambda_j k_j$, then we can induce that

$$T_{4k}\sum_{j=1}^{3} i(\lambda_j k_j)^2-\sum_{j=1}^{3} B_{jk}\lambda_j k_j=0, \ \ k\in\mathbb{Z}^3.$$

Hence we have

$$\begin{cases} T_{4,(0,0,0)}=a, \qquad a \text{ is an arbitrary constant}, \\ T_{4k}=\dfrac{\sum\limits_{j=1}^{3} k_j\lambda_j B_{jk}(t)}{\sum\limits_{j=1}^{3} i(k_j\lambda_j)^2}, \qquad k\in\mathbb{Z}^3\setminus\{0\}, \\ T_{jk}=\exp\left(-\sum\limits_{m=1}^{3}\nu(k_m\lambda_m)^2 t\right)\left(\int\limits_0^t (B_{jk}(s)-iT_{4k}(s)k_j\lambda_j)\exp\left(\sum\limits_{m=1}^{3}\nu(k_m\lambda_m)^2 s\right)\mathrm{d}s+A_{jk}\right), \\ \qquad\qquad j=1,2,3, \ \ k\in\mathbb{Z}^3. \end{cases}$$

Clearly we have:

Theorem 2.13. If

$$\sum_{k\in\mathbb{Z}^3}\sum_{m=1}^{3} k_m^2\left(|B_{jk}(t)|+|A_{jk}|\right)<+\infty, \quad t\geq 0, \ \ m=1,2,3, \tag{2.30}$$

then the series (2.23) we obtain is a solution of Eq. (2.22).

If $u_j(x,0)$, $f_j(x,t)$ $j=1,2,3$ are the real-valued functions, then we have $A_{jk}=\overline{A_{j,-k}}$, $B_{jk}(t)=\overline{B_{j,-k}(t)}$, $j=1,2,3$, $k\in\mathbb{Z}^3$. So we can induce that:

Theorem 2.14. If $u_j(x,0)$, $f_j(x,t)$ $j=1,2,3$ are the real-valued functions, then so do the functions (2.23) we obtain.

3 Series solutions to the cauchy problem for some more general LPDEs

In this section, using an iterative method with respect to (1.1), we deal with several LPDEs. This technique can solve many LPDEs.

Example 3.1.

$$\begin{cases} u_y - u_{xy} - \left(e^{e^{-(x+2)}} - 1\right) u = y e^{-(x+2)}, \\ (x,y) \in \Omega = \{(x,t) \mid x > 0,\ 0 \le y \le x\}, \\ u(x,0) = 1 + e^{-(x+2)}. \end{cases} \tag{3.1}$$

Clearly we have

$$\exp(e^{-(x+2)}) - 1 = \sum_{k \in \mathbb{N}_+} \frac{e^{-k(x+2)}}{k!}.$$

Next we set

$$u(x,y) = \sum_{k \in \mathbb{N}} T_k(y) e^{-k(x+2)}. \tag{3.2}$$

Suppose that the series (3.2) satisfies the following conditions:

$$\begin{cases} u = \sum\limits_{k \in \mathbb{N}} T_k(y) e^{-k(x+2)} \in C(\Omega), & (3.3) \\ u_t = \sum\limits_{k \in \mathbb{N}} T_k'(y) e^{-k(x+2)} \in C(\Omega), & (3.4) \\ u_{xt} = \sum\limits_{k \in \mathbb{N}_+} -k T_k'(y) e^{-k(x+2)} \in C(\Omega), & (3.5) \\ \left(e^{e^{-(x+2)}} - 1\right) u = \sum\limits_{k \in \mathbb{N}_+} \sum\limits_{m=0} k-1 \dfrac{T_m(y)}{(k-m)!} e^{-k(x+2)} \in C(\Omega). & (3.6) \end{cases}$$

Substituting the series (3.2) into the equations (3.1) we have

$$\begin{cases} T_0'(y) + (2T_1'(y) - T_0(y) - t) e^{-(x+2)} + \sum\limits_{k=2}^{+\infty} \left[(k+1) T_k'(y) - \sum\limits_{m=0}^{k-1} \frac{1}{(k-m)!} T_m(y) \right] e^{-k(x+2)} = 0, \\ u(x,0) = \sum\limits_{k \in \mathbb{N}} T_k(0) e^{-k(x+2)} = 1 + e^{-(x+2)}. \end{cases}$$

Let

$$\begin{cases} T_0'(t) = 0, & T_0(0) = 1, \\ 2T_1'(t) - T_0(t) - t = 0, & T_1(0) = 1, \\ (k+1) T_k'(t) - \sum\limits_{m=0}^{k-1} \frac{1}{(k-m)!} T_m(t) = 0, & T_k(0) = 0,\ k \ge 2. \end{cases}$$

Then we have

$$T_k(t) = \begin{cases} 1, & k = 0, \\ \frac{1}{4} t^2 + \frac{1}{2} t + 1, & k = 1, \\ \frac{1}{k+1} \int\limits_0^t \sum\limits_{m=0}^{k-1} \frac{1}{(k-m)!} T_m(s) \mathrm{d}s, & k \ge 2. \end{cases}$$

Next we prove that the formal solution (3.2) is also a solution of (3.1). By the induction method, we can prove that

$$0 < T_k(y) \le e^{ky}, \quad k \in \mathbb{N}.$$

So we have

$$0 < T_k(y) e^{-k(x+2)} \le e^{-k(x-y)-2k} \le e^{-2k}, \quad (x,t) \in \Omega,\ k \in \mathbb{N}.$$

Hence the series (3.2) converges uniformly on Ω. It means that the formal solution (3.2) satisfies (3.3). Moreover, we can prove that

$$\begin{cases} |T_k'(y)e^{-k(x+2)}| = \frac{1}{k+1}\left|\sum_{m=0}^{k-1}\frac{1}{(k-m)!}T_m(y)\right|e^{-k(x+2)} \leq e^{-2k}, & k\geq 2, \\ |-kT_k'(y)e^{-k(x+2)}| \leq ke^{-2k}, & k\geq 2, \\ \left|\sum_{m=0}^{k-1}\frac{1}{(k-m)!}T_m(y)e^{-k(x+2)}\right| \leq ke^{-2k}, & k\geq 2. \end{cases}$$

So the series (3.2) we obtain satisfies (3.4)-(3.6). Therefore it is a solution of (3.1).

Example 3.2.

$$\begin{cases} u_t+u+(x+3)^{\frac{2}{4}}u_x=0, \quad x\geq 0,\ t\geq 0, \\ u(x,0)=\sin\,(x+3)^{-\frac{1}{4}} = \sum\limits_{k\in\mathbb{N}_+}\frac{(-1)^{k+1}(x+3)^{-\frac{2k-1}{4}}}{(2k-1)!}. \end{cases} \tag{3.7}$$

Similar as Example 3.1, we can get a solution:

$$u(x,t)=\sum_{k\in\mathbb{N}_+}T_k(t)(x+3)^{-\frac{k}{4}},$$

where

$$T_k(t)=\begin{cases} e^{-t}, & k=1, \\ 0, & k=2,4,6,\cdots, \\ e^{-t}\left(\int_0^t\frac{k-2}{4}T_{k-2}(s)e^s\mathrm{d}s+\frac{(-1)^{\frac{k-1}{2}}}{k!}\right), & k=3,5,7,\cdots. \end{cases}$$

4 Series solutions to the cauchy problem for some NPDEs

In this section, similar as Section 3, using an iterative method with respect to (1.1), we deal with several NLPEs. This technique also can solve many NPDEs.

Lemma 4.1. (Abel identities [6]) For every $k\in\mathbb{N}_+$, we have

$$k(k+1)^k=\sum_{m=1}^{k}\binom{k+1}{m}m^m(k+1-m)^{k-m},$$

where $\binom{k+1}{m}=\frac{(k+1)!}{m!(k+1-m)!}$.

Example 4.2. (Inviscid Burgers' equation).

$$\begin{cases} u_t+uu_x=0,\ (x,t)\in\Omega=\{(x,t)\,|\,t\geq 0,\ x\in[0,11]\}, \\ u(x,0)=1+e^{x-12}. \end{cases} \tag{4.1}$$

Next we let

$$u(x,t)=\sum_{k\in\mathbb{N}}T_k(t)e^{k(x-12)}. \tag{4.2}$$

Suppose that the following conditions hold:

$$
\begin{cases}
u = \sum\limits_{k\in\mathbb{N}} T_k(t)e^{k(x-12)} \in C(\Omega), & (4.3)\\
u_t = \sum\limits_{k\in\mathbb{N}} T_k'(t)e^{k(x-12)} \in C(\Omega), & (4.4)\\
u_x = \sum\limits_{k\in\mathbb{N}_+} kT_k(t)e^{k(x-12)} \in C(\Omega), & (4.5)\\
uu_x = \sum\limits_{k\in\mathbb{N}_+} \sum\limits_{r=1}^{k} rT_r(t)T_{k-r}(t)e^{k(x-12)} \in C(\Omega). & (4.6)
\end{cases}
$$

Substituting the series (4.2) into (4.1), we get

$$
\begin{cases}
T_0' + (T_1' + T_0T_1)e^{x-12} + \sum\limits_{k=2}^{+\infty}(T_k' + kT_0T_k + \sum\limits_{r=1}^{k-1} rT_rT_{k-r})e^{k(x-12)} = 0,\\
u(x,0) = \sum\limits_{k\in\mathbb{N}} T_k(0)e^{k(x-12)} = 1 + e^{x-12}
\end{cases}
$$

Note that the sequence $\{e^{k(x-12)}\}_{k\in\mathbb{N}}$ is linearly independent, so we have

$$
\begin{cases}
T_0' = 0, & T_0(0) = 1,\\
T_1' + T_0T_1 = 0, & T_1(0) = 1,\\
T_k' + kT_0T_k + \sum\limits_{r=1}^{k-1} rT_rT_{k-r} = 0, & T_k(0) = 0,\ k \geq 2.
\end{cases}
$$

Then by Lemma 4.1, we can get

$$
T_k(t) = \begin{cases}
1, & k = 0,\\
e^{-t}, & k = 1,\\
e^{-kt}\int\limits_0^t \sum\limits_{r=1}^{k-1} -rT_r(s)T_{k-r}(s)e^{ks}\mathrm{d}s = (-1)^{k+1}\frac{k^{k-1}}{k!}t^{k-1}e^{-kt}, & k \geq 2.
\end{cases}
$$

So we get

$$
u(x,t) = 1 + e^{-t+x-12} + \sum_{k\geq 2}(-1)^{k+1}\frac{k^{k-1}}{k!}t^{k-1}e^{k(-t+x-12)}. \tag{4.7}
$$

Next we prove that the series (4.7) satisfies (4.3)-(4.6). Note that $\frac{k^m}{m!}t^m \leq e^{kt}$, $m,k \in \mathbb{N}$, so we have

$$
|T_k(t)e^{k(x-12)}| \leq \frac{1}{k}e^{k(x-12)} \leq \frac{1}{k}e^{-k}, \quad k \geq 2.
$$

So the series (4.7) converges uniformly on Ω. It means that the formal solution (4.7) satisfies (4.3). Moreover, we can prove that

$$
\begin{cases}
|\sum\limits_{r=1}^{k-1} rT_rT_{k-r}| = \frac{(k-1)k^{k-1}}{k!}t^{k-2}e^{-kt} = \frac{k^{k-2}}{(k-2)!}t^{k-2}e^{-kt} \leq 1, & k \geq 2,\\
|T_k'(t)e^{k(x-12)}| = |kT_0T_k + \sum\limits_{r=1}^{k-1} rT_rT_{k-r}|e^{k(x-12)} \leq 2e^{-k}, & k \geq 2,\\
|kT_k(t)e^{k(x-12)}| \leq e^{-k}, & k \geq 2.
\end{cases}
$$

Thus the series (4.7) is a solution of (4.1).

Example 4.3.

$$\begin{cases} u_t+(x+1)^2u_{xx}+u_xu=0, \quad x\geq 1,\ t\geq 0, \\ u(x,0)=(x+1)^{-1}+(x+1)^{-2}. \end{cases} \tag{4.8}$$

Similar as Example 4.2, we let

$$u(x,t)=\sum_{k\in\mathbb{N}_+} T_k(t)(x+1)^{-k}, \tag{4.9}$$

Suppose that the following conditions hold:

$$\begin{cases} u=\sum\limits_{k\in\mathbb{N}_+} T_k(t)(x+1)^{-k}\in C([1,+\infty)\oplus[0,+\infty)), & (4.10) \\ u_x=\sum\limits_{k\in\mathbb{N}_+} -kT_k(t)(x+1)^{-k-1}\in C([1,+\infty)\oplus[0,+\infty)), & (4.11) \\ u_t=\sum\limits_{k\in\mathbb{N}_+} T_k'(t)(x+1)^{-k}\in C([1,+\infty)\oplus[0,+\infty)), & (4.12) \\ u_{xx}=\sum\limits_{k\in\mathbb{N}_+} k(k+1)T_k(t)(x+1)^{-k-2}\in C([1,+\infty)\oplus[0,+\infty)), & (4.13) \\ u_xu=\sum\limits_{k\geq 3}\sum\limits_{r=1}^{k-2} -rT_r(t)T_{k-1-r}(t)(x+1)^{-k}\in C([1,+\infty)\oplus[0,+\infty)). & (4.14) \end{cases}$$

Substituting (4.9) into (4.8), we get

$$\begin{cases} (T_1'+2T_1)(x+1)^{-1}+(T_2'+6T_2)(x+1)^{-2}+\sum\limits_{k\geq 3}(T_k'+k(k+1)T_k \\ \qquad -\sum\limits_{r=1}^{k-2} rT_rT_{k-1-r})(x+1)^{-k}=0, \\ u(x,0)=\sum\limits_{k\in\mathbb{N}_+} T_k(0)(x+1)^{-k}=(x+1)^{-1}+(x+1)^{-2}. \end{cases}$$

Note that the sequence $\{(x+1)^{-k}\}_{k\in\mathbb{N}_+}$ is linearly independent, so we have

$$\begin{cases} T_1'+2T_1=0, & T_1(0)=1, \\ T_2'+6T_2=0, & T_2(0)=1, \\ T_k'+k(k+1)T_k-\sum\limits_{r=1}^{k-2} rT_rT_{k-r}=0, & T_k(0)=0,\ k\geq 3. \end{cases}$$

Then we get

$$T_k(t)=\begin{cases} e^{-2t}, & k=1, \\ e^{-6t}, & k=2, \\ e^{-k(k+1)t}\int\limits_0^t\sum\limits_{r=1}^{k-2} rT_r(s)T_{k-1-r}(s)e^{k(k+1)s}\mathrm{d}s, & k\geq 3. \end{cases}$$

Next we prove that the formal solution (4.9) satisfies (4.10)-(4.14). By the induction method, we can prove that

$$0<T_k(t)\leq e^{-(k+1)t}, \quad k\in\mathbb{N}_+. \tag{4.15}$$

So the series $\sum_{k\in\mathbb{N}_+} T_k(t)(x+1)^{-k}$ converges uniformly on $[1,+\infty)\oplus[0,+\infty)$. It means that the formal solution (4.9) satisfies (4.10). Moreover, we can prove that

$$\begin{cases} |-kT_k(t)(x+1)^{-k-1}| \leq ke^{-(k+1)t}(x+1)^{-k-1} \leq 2^{-k-1}k, & k\geq 3,\\ |T_k'(t)(x+1)^{-k}| = |k(k+1)T_k - \sum\limits_{r=1}^{k-2} rT_rT_{k-r}|(x+1)^{-k} \leq 2k(k+1)2^{-k}, & k\geq 3,\\ |k(k+1)T_k(t)(x+1)^{-k-2}| \leq k(k+1)2^{-k-2}, & k\geq 3,\\ |\sum\limits_{r=1}^{k-2} -rT_r(t)T_{k-1-r}(t)(x+1)^{-k}| \leq (k-2)(k-1)2^{-k}, & k\geq 3. \end{cases}$$

So the formal solution (4.9) satisfies (4.11)-(4.14). Thus it is a solution of (4.8).

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